Homework 2 Solutions – Relativity (`G0I36A`)

15/12/2020

1 Spherically Symmetric Spacetimes with a Cosmological Constant

1.) The Einstein equation in the presence of a cosmological constant Λ, but with no additional matter, is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = 0 \ . \tag{1.1}$$

Assuming that we are in a four-dimensional spacetime, the trace of this Einstein equation yields

$$R = 4\Lambda \ . \tag{1.2}$$

We can plug this into the original Einstein equation to eliminate R in favor of Λ. The result is simply

$$R_{\mu\nu} = \left(\frac{1}{2}R - \Lambda\right) g_{\mu\nu} = (2\Lambda - \Lambda) g_{\mu\nu} = \Lambda g_{\mu\nu} \ . \tag{1.3}$$

Thus we find a relation of the form given in the problem, with

$$\boxed{c = 1} \ . \tag{1.4}$$

2.) For a four-dimensional spherically symmetric spacetime, there always exists a coordinate frame in which the metric can be presented as

$$ds^2 = -e^{2\alpha(t,r)} dt^2 + e^{2\beta(t,r)} dr^2 + r^2 d\Omega^2 \ , \tag{1.5}$$

for some functions α and β. This is discussed in section 5.2 of Carroll, and I prove this explicitly in appendix A. For the purposes of the homework, though, it's fine if you simply start from this metric. The Ricci tensor components for the metric (1.5) are given in equation (5.41) in section 5.2 of Carroll, so we will make our life easier by simply using their results:

$$\begin{aligned}
R_{tt} &= \left[\partial_t^2 \beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] + e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \ , \\
R_{rr} &= e^{2(\beta-\alpha)}\left[\partial_t^2 \beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] - \left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\right] \ , \\
R_{tr} &= \frac{2}{r}\partial_t\beta \ , \\
R_{\theta\theta} &= e^{-2\beta}\left(r\partial_r\beta - r\partial_r\alpha - 1\right) + 1 \ , \\
R_{\phi\phi} &= \sin^2\theta \, R_{\theta\theta} \ ,
\end{aligned} \tag{1.6}$$

with all other components vanishing. Note that implicitly α and β depend on t and r, but we have suppressed this dependence for notational simplicity.

Our goal in this problem is to solve the Einstein equation $R_{\mu\nu} = \Lambda g_{\mu\nu}$ from the previous problem for this spherically symmetric metric in order to determine the functions α and

β. First, we note that the metric is diagonal and so the (t, r) component of the Einstein equation tells us that $R_{tr} = 0$, and thus

$$\partial_t \beta = 0 \ . \tag{1.7}$$

Then, we take the (θ, θ) component of the Einstein equation, which tells us that

$$e^{-2\beta} \left(r\partial_r \beta - r\partial_r \alpha - 1 \right) + 1 = \Lambda r^2 \ . \tag{1.8}$$

If we differentiate this entire expression with respect to t and use the earlier-derived fact that $\partial_t \beta = 0$, we find simply that

$$\partial_t \partial_r \alpha = 0 \ . \tag{1.9}$$

Note that we used the fact that partial derivatives commute in deriving this, so that $\partial_t \partial_r \beta = \partial_r \partial_t \beta = 0$.

Our results (1.7) and (1.9) can be solved by writing the functions α and β as

$$\alpha(t, r) = f(r) + g(t) \ , \quad \beta(t, r) = h(r) \ , \tag{1.10}$$

for some functions f, g and h. Plugging this into our metric (1.5), this means that our metric takes the form

$$ds^2 = -e^{2f(r)} e^{2g(t)} dt^2 + e^{2h(r)} dr^2 + r^2 d\Omega^2 \ . \tag{1.11}$$

The time dependence in this can be simplified by means of a coordinate transformation. In particular, we will choose a new time coordinate $\tilde{t}$ defined such that

$$d\tilde{t} = e^{g(t)} dt \ , \tag{1.12}$$

which in turn lets us write the metric as

$$ds^2 = -e^{2f(r)} d\tilde{t}^2 + e^{2h(r)} dr^2 + r^2 d\Omega^2 \ . \tag{1.13}$$

We can then simply relabel the time coordinate $\tilde{t} \to t$, as well as relabelling the metric functions by $f \to \alpha$ and $h \to \beta$. We therefore have by means of a coordinate transformation of (1.11) that the metric can always be written as

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \ . \tag{1.14}$$

This is precisely the same as our starting metric (1.5), except the functions α and β no longer have time dependence. Therefore the Einstein equation demands that the metric is in fact *static*, and hence equipped with a timelike Killing vector. The components of the Ricci tensor hence simplify drastically:

$$\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)} \left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha \right] \ , \\
R_{rr} &= -\left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta - \frac{2}{r} \partial_r \beta \right] \ , \\
R_{\theta\theta} &= e^{-2\beta} \left(r\partial_r \beta - r\partial_r \alpha - 1 \right) + 1 \ , \\
R_{\phi\phi} &= \sin^2 \theta \, R_{\theta\theta} \ ,
\end{aligned} \tag{1.15}$$

with all other components vanishing.

Now, we need to solve the remaining components of the Einstein equation. First, we will take a particular linear combination of the (t, t) and (r, r) components:

$$e^{2(\beta-\alpha)} R_{tt} + R_{rr} = \Lambda e^{2(\beta-\alpha)} g_{tt} + \Lambda g_{rr} \ . \tag{1.16}$$

The particular combination we chose makes the left-hand side simplify drastically, and we are left with the relationship

$$\frac{2}{r}\left(\partial_r\alpha + \partial_r\beta\right) = \Lambda e^{2(\beta-\alpha)}\left(-e^{2\alpha}\right) + \Lambda\left(e^{2\beta}\right) = 0 \ . \tag{1.17}$$

That is, the left-hand side becomes proportional to a single derivative, and the right-hand side vanishes. We thus find that $\alpha'(r) = -\beta'(r)$, which in turn can be solved by setting $\alpha(r) = -\beta(r) + c$ for some constant c. We can without loss of generality choose $c = 0$, though, because c will simply correspond to a constant rescaling of the time coordinate, and we can always remove such a scaling by doing a coordinate transformation. We can therefore, without loss of generality, set

$$\beta(r) = -\alpha(r) \ , \tag{1.18}$$

leaving us with only one function $\alpha(r)$ left to determine.

To determine it, we will look at the (θ,θ) component of the Einstein equation $R_{\theta\theta} = \Lambda g_{\theta\theta}$, which can now be written explicitly as

$$-\,e^{2\alpha}\left(2r\partial_r\alpha + 1\right) + 1 = \Lambda r^2 \ . \tag{1.19}$$

This is a first order differential equation that we can rewrite as

$$\frac{d}{dr}\left(re^{2\alpha}\right) = 1 - \Lambda r^2 \ . \tag{1.20}$$

This is easily integrated to find that the most general solution is

$$e^{2\alpha} = 1 + \frac{C}{r} - \frac{\Lambda}{3}r^2 \ , \tag{1.21}$$

where C is some integration constant. In the $\Lambda = 0$ limit, this should match the Schwarzschild solution, which fixes the integration constant to be $C = -2GM$, where G is Newton's constant and M is the black hole mass. The end result is that the most general spherically symmetric solution in the presence of a cosmological constant Λ is

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2 d\Omega^2} \ . \tag{1.22}$$

3.) This calculation pretty much just follows section 5.4 in Carroll, except with the additional cosmological constant term in the metric above. The time-like Killing vector $K = \partial_t$ associated with energy conservation has components

$$K^\mu = (1,0,0,0) \ , \quad K_\mu = \left(-\left(1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2\right), 0, 0, 0\right) \ . \tag{1.23}$$

The conserved energy is

$$E = -K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2\right)\frac{dt}{d\lambda} \ . \tag{1.24}$$

The Killing vector $R = \partial_\phi$ associated with conservation of the z-component of the angular momentum has components

$$R^\mu = (0,0,0,1) \ , \quad R_\mu = (0,0,0,r^2\sin^2\theta) \ . \tag{1.25}$$

We will judiciously choose our coordinates such that the geodesic is fixed at $\theta = \frac{\pi}{2}$. We are allowed to do this without loss of generality due to the presence of two other Killing vectors on our background that correspond to conservation of the x- and y-components of the angular momentum as well. Thus, the magnitude L of the angular momentum is given by

$$L = R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda} \ . \tag{1.26}$$

With the conserved quantities E and L written down, we now recall that the quantity

$$\epsilon = -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{1.27}$$

is also conserved along geodesics. We choose the affine parameter λ such that ϵ is normalized to be $\epsilon = 1$ for time-like geodesics and $\epsilon = 0$ for null gedoesics. Thus, massive particle trajectories and massless partile trajectories will have $\epsilon = 1$ and $\epsilon = 0$, respectively.

Writing ϵ out explicitly and recalling that θ is fixed to $\frac{\pi}{2}$, we find

$$\epsilon = \left(1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2\right) \left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2\right)^{-1} \left(\frac{dr}{d\lambda}\right)^2 - r^2 \left(\frac{d\phi}{d\lambda}\right)^2 \ . \tag{1.28}$$

We can use (1.24) and (1.26) to replace $\frac{dt}{d\lambda}$ and $\frac{d\phi}{d\lambda}$ with E and L, respectively, along with some functions of r. The result of this is that we can rewrite this expression as

$$\frac{1}{2} \left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E} \ , \tag{1.29}$$

where the effective potential $V(r)$ is given by

$$V(r) = \frac{\epsilon}{2} - \epsilon \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} - \epsilon \frac{\Lambda}{6} r^2 - \frac{\Lambda}{6} L^2 \ , \tag{1.30}$$

and the effective "energy" $\mathcal{E}$ is given by

$$\mathcal{E} = \frac{E^2}{2} \ . \tag{1.31}$$

We can plot the potential $V(r)$ for different values of the cosmological constant Λ in order to get a feel for how things change when $\Lambda > 0$ and when $\Lambda < 0$. We will choose units in which $GM = 1$ for simplicity for the rest of this problem.

In figures 1 and 2, I plot the potential for a massless and massive particle, respectively, for the case of $\Lambda = 0$. There are no stable orbits for the massless case, though there can be some for the massive case when L is large enough. In both cases, the potential asymptotes to a constant value, either 0 or 1/2 depending on whether the particle is massless or massive.

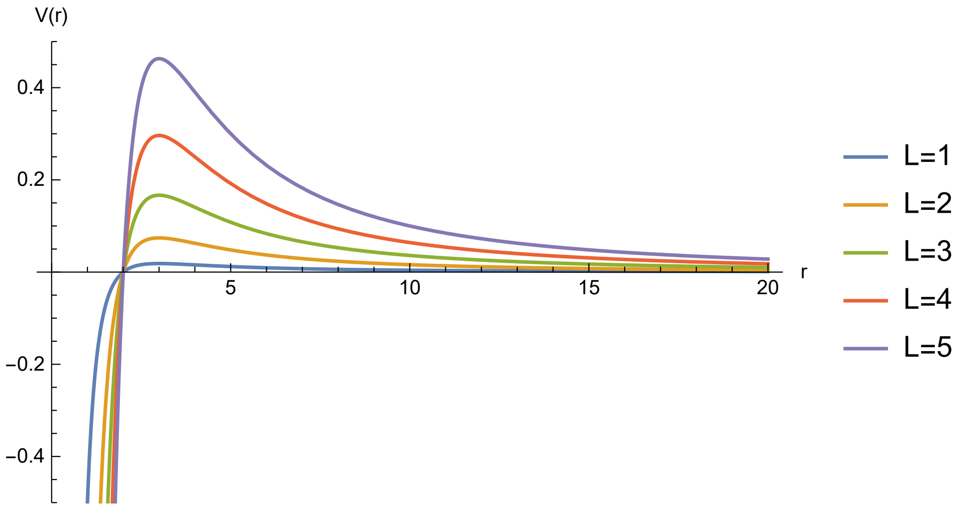

Figure 1. The effective potential $V(r)$ for a massless particle with $\Lambda = 0$ and various values of L, in units where $GM = 1$.

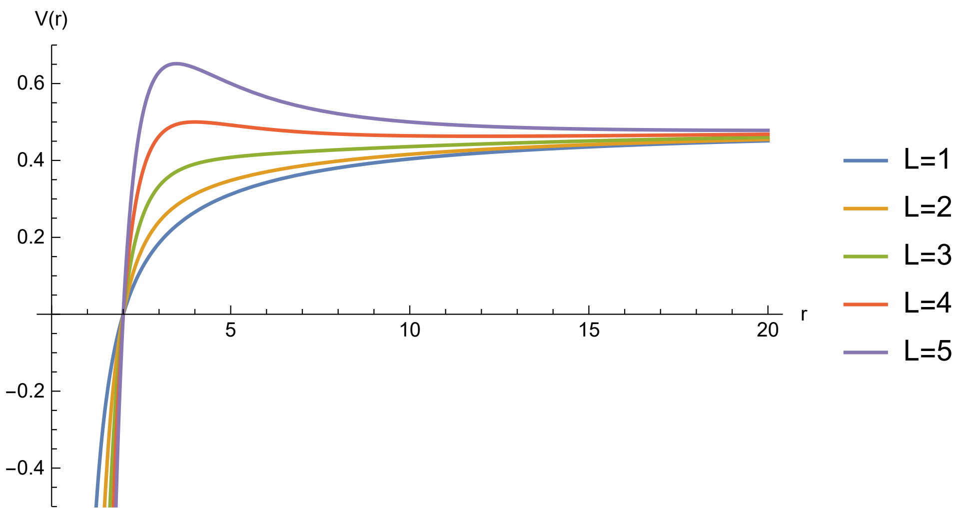

Figure 2. The effective potential $V(r)$ for a massive particle with $\Lambda = 0$ and various values of L, in units where $GM = 1$.

In figures 3 and 4, I plot the potential for a massless and massive particle, respectively, for the case of $\Lambda = 0.5$. Both cases have a negative potential everywhere, in contrast to how the $\Lambda = 0$ potential was positive everywhere outside the horizon radius. Additionally, while the massless potential asymptotes to a constant value just as it would for the $\Lambda = 0$ case, the massive potential goes like $\sim -\Lambda r^2$ for large values of r, and so the potential diverges negatively. This indicates that particles in unbound orbits that escape will actually *accelerate* as they get further from the black hole, no matter what their total energy is.

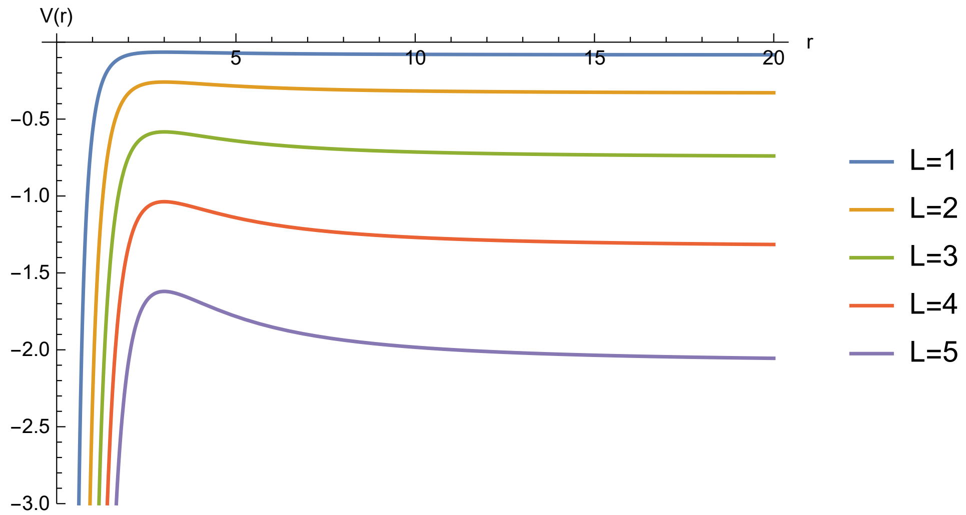

Figure 3. The effective potential $V(r)$ for a massless particle with $\Lambda = 0.5$ and various values of L, in units where $GM = 1$.

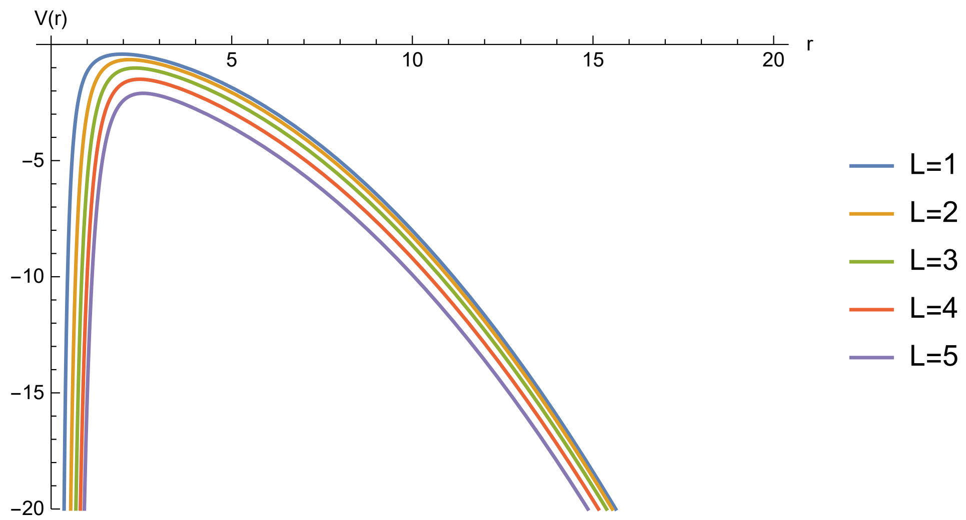

Figure 4. The effective potential $V(r)$ for a massive particle with $\Lambda = 0.5$ and various values of L, in units where $GM = 1$.

In figures 5 and 6, I plot the potential for a massless and massive particle, respectively, for the case of $\Lambda = -0.5$. The massless potential is positive outside the horizon radius and asymptotes to a constant value, just as it would for the $\Lambda = 0$ case. However, the massive potential goes like $\sim \Lambda r^2$ for large values of r, and so the potential diverges positively. This indicates that there are no unbound orbits for massive particles when the cosmological constant is negative; no matter what their starting energy, there will always be a turning point eventually as the particle travels to greater and greater values of r.

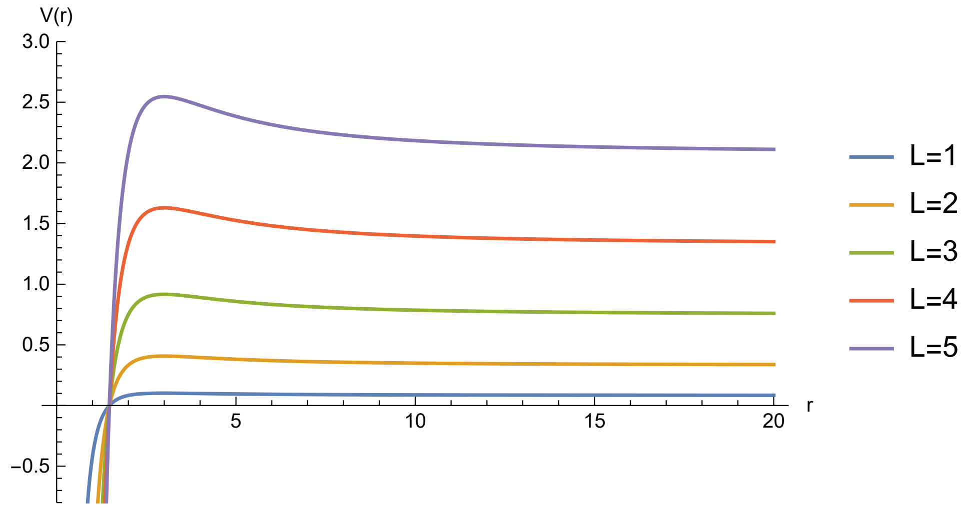

Figure 5. The effective potential $V(r)$ for a massless particle with $\Lambda = -0.5$ and various values of L, in units where $GM = 1$.

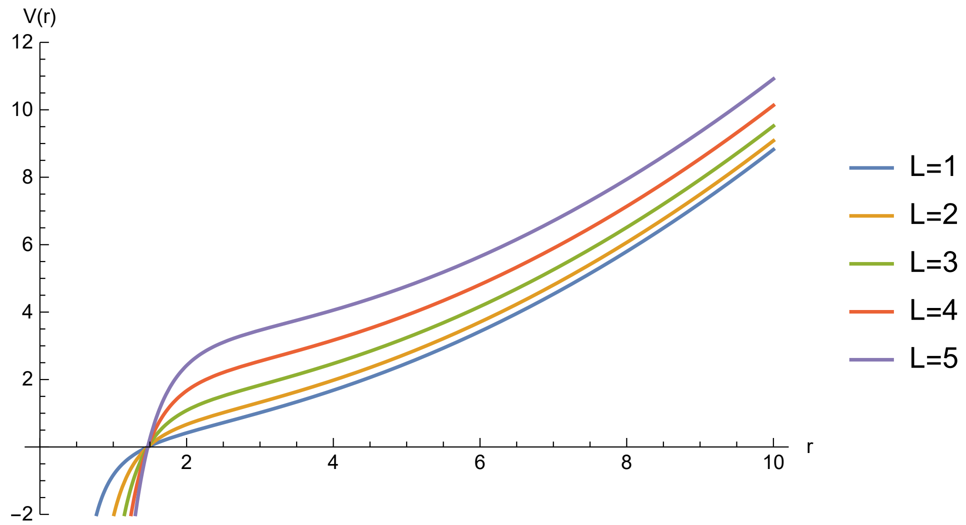

Figure 6. The effective potential $V(r)$ for a massive particle with $\Lambda = -0.5$ and various values of L, in units where $GM = 1$.

Lastly, I should note that we have not explicitly analyzed where the stable and unstable circular orbits are for these potentials. Doing so would be a rather involved and arduous task, since solving for the locations in which $V'(r) = 0$ requires solving a quintic polynomial equation when $\Lambda \neq 0$. Moreover, the location of the circular orbits depends heavily on how large the angular momentum L and cosmological constant Λ are relative to GM, and so there is a huge parameter space we would have to explore. So, even if the process for determining circular orbits is in some sense algorithmic, it's not worth the time and effort on this homework assignment.

2 Energy and Angular Momentum for Charged Particles

For a particle of charge q and mass m moving in the presence of an electromagnetic field, the geodesic equation is modified, simply because there is now a force other than gravity acting on the object. The modified geodesic equation takes the form

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = \frac{q}{m}F^\mu{}_\nu\frac{dx^\nu}{d\lambda}\ , \tag{2.1}$$

where λ is the affine parameter along the geodesic and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the electromagnetic field strength corresponding to the Maxwell field A_μ. This geodesic equation can also be written equivalently as

$$\frac{D}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = \frac{q}{m}F^\mu{}_\nu\frac{dx^\nu}{d\lambda}\ , \tag{2.2}$$

where $\frac{D}{d\lambda} \equiv \frac{dx^\mu}{d\lambda}\nabla_\mu$ denotes the covariant derivative with respect to the affine parameter λ.

In the absence of the background electromagnetic field, we know that the quantity $K_\mu\frac{dx^\mu}{d\lambda}$ should be constant along the geodesic for any Killing vector K. However, due to the modification of the geodesic equation by the electromagnetic field, we can now see that this is no longer conserved:

$$\begin{aligned}\frac{D}{d\lambda}\left(K_\mu\frac{dx^\mu}{d\lambda}\right) &= \frac{dx^\nu}{d\lambda}\nabla_\nu\left(K_\mu\frac{dx^\mu}{d\lambda}\right)\\ &= \frac{dx^\nu}{d\lambda}(\nabla_\nu K_\mu)\frac{dx^\mu}{d\lambda} + \frac{dx^\nu}{d\lambda}K_\mu\left(\nabla_\nu\frac{dx^\mu}{d\lambda}\right)\\ &= \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\nabla_\nu K_\mu + K_\mu\frac{D}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right)\\ &= \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\nabla_{(\nu}K_{\mu)} + K_\mu\left(\frac{q}{m}F^\mu{}_\nu\frac{dx^\nu}{d\lambda}\right)\\ &= \frac{q}{m}K^\mu F_{\mu\nu}\frac{dx^\nu}{d\lambda}\ ,\end{aligned} \tag{2.3}$$

where in the second line we used the product rule, in the third line we used the definition of the covariant derivative with respect to λ and reordered terms, in the fourth line we used the fact that $x^\mu x^\nu$ is symmetric under $\mu \leftrightarrow \nu$ as well as our geodesic equation (2.2), and then in the last line we used the Killing equation $\nabla_{(\mu}K_{\nu)} = 0$. Thus, we have shown that $K_\mu\frac{dx^\mu}{d\lambda}$ is no longer conserved along the geodesic.

So, we need to find some new quantity that can compensate for this non-conservation. In particular, since the last line of (2.3) involves both the Killing vector K as well as the Maxwell field A_μ, let's investigate how the product of these two vectors $A_\mu K^\mu$ changes along the geodesic:

$$\begin{aligned}\frac{D}{d\lambda}(A_\mu K^\mu) &= \frac{dx^\nu}{d\lambda}\nabla_\nu(A_\mu K^\mu)\\ &= \frac{dx^\nu}{d\lambda}\Big[(\nabla_\nu A_\mu)K^\mu + A_\mu\nabla_\nu K^\mu\Big]\\ &= \frac{dx^\nu}{d\lambda}\Big[(\nabla_\nu A_\mu - \nabla_\mu A_\nu)K^\mu + (\nabla_\mu A_\nu)K^\mu + A_\mu\nabla_\nu K^\mu\Big]\\ &= \frac{dx^\nu}{d\lambda}\Big[F_{\nu\mu}K^\mu + K^\mu\nabla_\mu A_\nu + A^\mu\nabla_\nu K_\mu\Big]\ ,\end{aligned} \tag{2.4}$$

where in the second line we used the product rule, in the third line we added and subtracted the same term in order to give us an antisymmetric derivative acting on the Maxwell field, and in the last line we recognized electromagnetic field strength and reorganized terms somewhat. From here, we can use antisymmetry of the field strength $F_{\nu\mu} = -F_{\mu\nu}$ as well as the Killing vector equation $\nabla_\nu K_\mu = -\nabla_\mu K_\nu$ to rewrite this as

$$\frac{D}{d\lambda}(A_\mu K^\mu) = -K^\mu F_{\mu\nu}\frac{dx^\nu}{d\lambda} + \frac{dx^\nu}{d\lambda}(K^\mu\nabla_\mu A_\nu - A^\mu\nabla_\mu K_\nu)\ . \tag{2.5}$$

Thus, we have exactly the kind of field strength term we need to cancel the one in (2.3) by also multiplying by an overall factor of q/m. We thus find that

$$\frac{D}{d\lambda}\left(K_\mu \frac{dx^\mu}{d\lambda} + \frac{q}{m} A_\mu K^\mu\right) = \frac{dx^\nu}{d\lambda}\left(K^\mu \nabla_\mu A_\nu - A^\mu \nabla_\mu K_\nu\right) \ . \tag{2.6}$$

So, we still don't quite have something that's conserved along the geodesic, but we're getting closer. What we want to do now is notice that

$$K^\mu \nabla_\mu A^\nu - A^\mu \nabla_\mu K^\nu = K^\mu \partial_\mu A^\nu - A^\mu \partial_\mu K^\nu \ , \tag{2.7}$$

as the symmetry in the lower indices of the Christoffel symbols forces the two to cancel when we expand the covariant derivatives. Then, we recognize (2.7) simply as the definition of the vector commutator $[K, A]^\nu$. Plugging this back into (2.6), we thus find that

$$\frac{D}{d\lambda}\left(K_\mu \frac{dx^\mu}{d\lambda} + \frac{q}{m} A_\mu K^\mu\right) = g_{\mu\nu} \frac{dx^\mu}{d\lambda}[K, A]^\nu \ . \tag{2.8}$$

This result tells us that if our background is equipped with a Killing vector K for which the commutator $[K, A]$ vanishes, then the right-hand side of (2.8) vanishes and thus the quantity

$$K_\mu \left(\frac{dx^\mu}{d\lambda} + \frac{q}{m} A^\mu\right) \tag{2.9}$$

will be conserved along any geodesic. That is, for any Killing vector K that commutes with A_μ, there is an associated conserved quantity.

What we have done so far holds in any arbitrary spacetime with an electromagnetic field turned on. We will now specialize to the four-dimensional Reissner-Nordström spacetime, for which the metric and the Maxwell field are given by

$$\begin{aligned} &ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ , \quad f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \ , \\ &A = \frac{Q}{r} dt \ . \end{aligned} \tag{2.10}$$

As discussed in class and in exercise sessions, this spacetime is equipped with four independent Killing vectors:

$$\begin{aligned} K_1 &= \partial_t \ , \\ K_2 &= -\sin\phi\, \partial_\theta - \cot\theta \cos\phi\, \partial_\phi \ , \\ K_3 &= \cos\phi\, \partial_\theta - \cot\theta \sin\phi\, \partial_\phi \ , \\ K_4 &= \partial_\phi \ . \end{aligned} \tag{2.11}$$

For uncharged particles moving along geodesics, we know that K_1 corresponds to energy conservation, while K_2, K_3, and K_4 correspond to conservation of (the three components of) angular momentum. To see if they are still associated with conserved quantities for charged geodesics, we need to determine if they commute with A_μ. To do so, we first note that

$$A^\mu = g^{\mu\nu} A_\nu = \left(-\frac{Q}{f(r)r}, 0, 0, 0\right) \ , \tag{2.12}$$

and so A^μ has only a time component, and this time component depends only on r. Since all of the Killing vectors above have no radial component, this means that $K^\mu \partial_\mu$ will involve no derivatives with respect to r, and thus

$$K^\mu \partial_\mu A^\nu = 0 \ . \tag{2.13}$$

Additionally, since none of the Killing vectors above have any component of K^μ that depends on the time coordinate t, this means that $\partial_t K^\mu = 0$. Then, since A^μ has only a time component, we thus conclude that

$$A^\mu \partial_\mu K^\nu = 0 \ . \tag{2.14}$$

Putting these two results together, we therefore find that $[K, A] = 0$ for all four Killing vectors on our Reissner-Nordström background. Thus each of them corresponds to a conserved charge!

We will use the conservation associated with K_2 and K_3 to fix the angular momentum to point purely in the z-direction, i.e. we will fix $\theta = \pi/2$ along the geodesic. Once we have done so, we now want to compute the total conserved energy associated with $K_1 = \partial_t$ and the total (magnitude of the) conserved angular momentum associated with $K_4 = \partial_\phi$.

First, we focus on $K_1 = \partial_t$, or equivalently $(K_1)^t = 1$, and thus $(K_1)_t = g_{tt} = -f(r)$. The conserved energy is thus given by, with analogy to how we computed it for the Schwarzschild solution,

$$\begin{aligned} E &= -m \, (K_1)_\mu \left(\frac{dx^\mu}{d\lambda} + \frac{q}{m} A^\mu \right) \\ &= m f(r) \left(\frac{dt}{d\lambda} + \frac{q}{m} A^t \right) \\ &= m f(r) \frac{dt}{d\lambda} - \frac{qQ}{r} \\ &= \left(m - \frac{2GMm}{r} + \frac{GQ^2 m}{r^2} \right) \frac{dt}{d\lambda} - \frac{qQ}{r} \ . \end{aligned} \tag{2.15}$$

We put an overall factor of $-m$ in front of the conserved quantity for convenience, and to make the units match those of energy, but of course the quantity is conserved whether or not you have these additional factors. Notice that the sign of the electromagnetic interaction energy depends on the relative sign between the particle charge q and the black hole charge Q. In the Schwarzschild limit $Q \to 0$, this becomes

$$E|_{\text{Schw.}} = \left(m - \frac{2GMm}{r} \right) \frac{dt}{d\lambda} \ . \tag{2.16}$$

Thus there are two new features of the energy in the Reissner-Nordström background. The presence of the black hole charge Q modifies the metric itself which in turn changes the energy associated with the gravitational field. Additionally, the interaction between the particle charge and black hole charge also causes an additional contribution to the energy that goes like $1/r$, as expected from Coulomb's law.

Now, let's move on to the Killing vector $K_4 = \partial_\phi$, or equivalently $(K_4)^\phi = 1$ and thus $(K_4)_\phi = g_{\phi\phi} = r^2 \sin^2\theta = r^2$, since we have fixed $\theta = \pi/2$. The conserved angular momentum L is thus given by

$$\begin{aligned} L &= (K_4)_\mu \left(\frac{dx^\mu}{d\lambda} + \frac{q}{m} A^\mu \right) \\ &= r^2 \left(\frac{d\phi}{d\lambda} + \frac{q}{m} A^\phi \right) \\ &= r^2 \frac{d\phi}{d\lambda} \ . \end{aligned} \tag{2.17}$$

We get exactly the same thing in the Schwarzschild limit $Q \to 0$, and thus the angular momentum has the same form for any value of the charge Q. Thus, the form of the angular momentum seems to be unmodified by the presence of charge. Intuitively, this is because both the electromagnetic force and the gravitational force act purely in the radial direction and so the angular momentum does not receive any modifications from its usual form. The only thing that is relevant is the rate $d\phi/d\lambda$ at which the particle is orbiting the black hole, as well as the distance r to the black hole.

A Spherical Symmetry and Coordinate Transformations

In four dimensions, the most general spherically symmetric metric takes the form

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + e^{2\gamma(t,r)}d\Omega^2 \ , \tag{A.1}$$

for some functions α, β, and γ, and some coordinates $x^\mu = (t, r, \theta, \phi)$. Of course, we are free to make coordinate transformations to simplify this metric. In particular, we will show in this appendix that there exists a series of coordinate transformations through which the metric can always be rewritten as

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\Omega^2 \ , \tag{A.2}$$

for some functions α and β, though we should note that these are not necessarily the same functions as in (A.1).

The first step is to define a new coordinate R by

$$R \equiv e^{\gamma(t,r)} \ . \tag{A.3}$$

Using the chain rule, this then tells us that the differentials dR, dt, and dr are related by

$$dR = (\partial_t\gamma(t,r))\, e^{\gamma(t,r)}dt + (\partial_r\gamma(t,r))\, e^{\gamma(t,r)}dr \ . \tag{A.4}$$

Assuming that $\partial_r\gamma(t,r) \neq 0$,[1] we can solve for dr explicitly as

$$dr = -\left(\frac{\partial_t\gamma(t,r)}{\partial_r\gamma(t,r)}\right) dt + \left(\frac{e^{-\gamma(t,r)}}{\partial_r\gamma(t,r)}\right) dR \ . \tag{A.5}$$

We can use this to eliminate dr in the metric (A.1) in favor of dt and dR, as well as trade the $e^{2\gamma(t,r)}$ term in front of the S^2 metric for R^2. The result of this procedure is

$$\begin{aligned} ds^2 = &-\left(e^{2\alpha(t,r)} - \left(\frac{\partial_t\gamma(t,r)}{\partial_r\gamma(t,r)}\right)^2 e^{2\beta(t,r)}\right) dt^2 + \left(\frac{e^{2\beta(t,r)-2\gamma(t,r)}}{(\partial_r\gamma(t,r))^2}\right) dR^2 \\ &- 2\left(\frac{\partial_t\gamma(t,r)e^{2\beta(t,r)-\gamma(t,r)}}{(\partial_r\gamma(t,r))^2}\right) dt\, dR + R^2 d\Omega^2 \ . \end{aligned} \tag{A.6}$$

Notice that the coordinate r is implicitly a function of R and t via equation (A.3), which in turn means that all three terms in parentheses in (A.6) are implicitly functions of t and R. Thus, there exist some functions A, B, and C of the coordinates t and R such that this metric can be written as

$$ds^2 = -e^{2A(t,R)}dt^2 + e^{2B(t,R)}dR^2 - 2e^{2C(t,R)}dt\, dR + R^2 d\Omega^2 \ . \tag{A.7}$$

Notice that this form of the metric is not diagonal, due to the cross-terms proportional to $dt\, dR$. Our goal now is to find a new coordinate $T(t,R)$ such that we eliminate t in favor of T and end up with a diagonal metric. That is, we want the metric to take the form

$$ds^2 = -e^{2f(T,R)}dT^2 + e^{2g(T,R)}dR^2 + R^2 d\Omega^2 \ , \tag{A.8}$$

[1] If we had $\partial_r\gamma(t,r) = 0$, then instead of trading dr for dt and dR, we could instead just trade dt for dR and dr, and then the rest of the derivation would carry through in similar fashion. So, we can make this assumption with impunity.

for some new functions $f(T,R)$ and $g(T,R)$. The chain rule tells us that dT is given in terms of dt and dR by

$$dT = \frac{\partial T}{\partial t}dt + \frac{\partial T}{\partial R}dR \ , \tag{A.9}$$

and thus our goal metric (A.8) can be written as

$$\begin{aligned} ds^2 = &-\left(\frac{\partial T}{\partial t}\right)^2 e^{2f(T,R)}dt^2 - 2\left(\frac{\partial T}{\partial t}\right)\left(\frac{\partial T}{\partial R}\right)e^{2f(T,R)}dt\, dR \\ &+\left(e^{2g(T,R)} - \left(\frac{\partial T}{\partial R}\right)^2 e^{2f(T,R)}\right)dR^2 + R^2 d\Omega^2 \ . \end{aligned} \tag{A.10}$$

If we want this goal metric (A.10) to be equal to our actual metric (A.7), we can compare the coefficients of dt^2, dR^2, and $dt\, dR$ to obtain the following three conditions:

$$\begin{aligned} e^{2A(t,R)} &= \left(\frac{\partial T}{\partial t}\right)^2 e^{2f(T,R)} \ , \\ e^{2B(t,R)} &= e^{2g(T,R)} - \left(\frac{\partial T}{\partial R}\right)^2 e^{2f(T,R)} \ , \\ e^{2C(t,R)} &= \left(\frac{\partial T}{\partial t}\right)\left(\frac{\partial T}{\partial R}\right)e^{2f(T,R)} \ . \end{aligned} \tag{A.11}$$

That is, we need some new functions T, f, and g that are related to the old functions A, B, and C in precisely the way prescribed in (A.11). This must always be possible, though, because we have precisely three constaints and we want three new functions, and so we have exactly as many unknowns as we do constaint equations. Thus it must always be possible to find some functions T, f, and g to solve (A.11). Thus we will inevitably be able to put our metric in the form of (A.8).

The upshot of this discussion is that, through a series of coordinate transformations, we can take our starting spherically symmetric spacetime (A.1) and always end up with a metric of the form (A.8). Finally, since T and R are just labels for coordinates, while f and g are just labels for functions, there is nothing stopping us from renaming them. In particular, we can now make the following redefinitions:

$$\begin{aligned} T &\to t \ , \\ R &\to r \ , \\ f &\to \alpha \ , \\ b &\to \beta \ . \end{aligned} \tag{A.12}$$

With these redefinitions, the metric will take the form

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\Omega^2 \ , \tag{A.13}$$

This is exactly what we wrote in (A.2), and is of the same form as the starting metric (A.1), except the function $e^{2\gamma(t,r)}$ has been replaced with r^2. Thus, we have established that there is a coordinate transformation from the original metric (A.1) to (A.2) that fixes the function in front of the S^2 metric to just be the usual r^2, which makes our work in problem 1 easier.